State Abstraction Discovery in Model-Based Reinforcement Learning

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Markov Decision Processes

- Observable State s_t, Action a_t, Reward r_t, Next state s_{t+1}
- Optimization problem : $\max_{\pi \in \mathcal{A}^{\mathcal{S}}} \sum_{t \geq 0} \gamma^t r_t$, $\gamma = 0.99$

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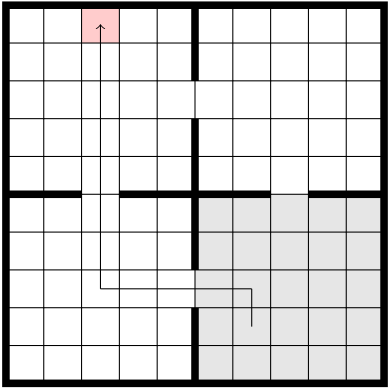
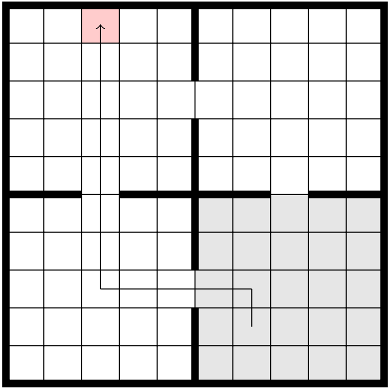
Four Rooms instance



Four rooms:

- $\mathcal{S} = [\![1, 100]\!]$, $\mathcal{A} = \{N, S, E, W\}$
- Reward: -1 until exit is reached, 0 otherwise.
- Step forward success with probability .8 (if step is doable)

Four Rooms instance



Naturally hierarchical!

- Spatial: each room as a single state
- Temporal: learn to exit a room

Hierarchical Reinforcement Learning

Why use HRL in MDPs?

- Solving large MDPs
- Enhancing explainability and interpretability of MDPs
- Ensuring solution quality

How to implement it?

- Subgoal discovery and meta-action learning (temporal abstraction)
- Building approximate MDPs using spatial abstraction, with limited explicit methods

Related works

Handling Large MDP Spaces:

- With factorization hypothesis[1]
- Hierarchical approach[2]
- State information abstraction[3]

State aggregation and abstraction discovery:

- Abstraction discovery[4]
- Abstraction and MDP approximation[5]
- MDP solving acceleration[6]

[1] [Guestrin et al., 2003, Siddiqi et al., 2010]

[2] [Sutton et al., 1999, Li et al., 2006, Hengst, 2012]

[3] [Pineau et al., 2003, Coulom, 2006]

[4] [Singh et al., 1994, Dean and Givan, 1997, Abel et al., 2016, Ferrer-Mestres et al., 2020]

[5] [Tsitsiklis and Van Roy, 1996, Abel, 2019, Gopalan et al., 2017]

[6] [Bean et al., 1987, Bertsekas et al., 1988, Ciosek and Silver, 2015, Abel et al., 2020, Jothimurugan et al., 2021]

Context and work

In this talk, we present:

- MDP solving context
- The HRL approach
- Our contribution[7]

In state abstraction, we:

- Link MDP abstraction and Approximate Dynamic Programming
- Estimate error induced by abstraction
- Present a practical way to abstract MDP and solve them exactly
- Conduct a numerical comparison using real-world models

[7] [Forghieri et al., 2024]

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MDP solving approach

$$
\begin{aligned}
\text{Solving the MDP} &\iff \text{Maximizing upcoming rewards relatively to } \pi \\
&\iff \max_{\text{policy}} \mathop{\mathbb{E}}_{a_t = \pi(s_t)} \left[\sum_{t \geq 0} \gamma^t r_t \,\middle|\, s_0 = s \right] \\
&\iff \max_{\pi \in \mathcal{A}^{\mathcal{S}}} V^\pi(s) := V^*(s)
\end{aligned}
$$

Value functions definitions

- The expected reward applying π, V^π:

$$V^\pi(s) = \underset{a_t = \pi(s_t)}{\mathbb{E}} \left[\sum_{t \geq 0} \gamma^t r_t \middle| s_0 = s \right]$$

- The optimal value function:

$$V^*(s) = \max_\pi V^\pi(s)$$

Moreover, V^* is solution of the optimal Bellman equation (fixed point equation):

$$V^* = \max_{a \in \mathcal{A}} (R_a + \gamma \cdot T_a \cdot V^*) := \mathcal{T}^* V$$

How to find V^* or π^* ?

Dynamic Programming approaches[8]:

- Value Iteration

$$V_{t+1} \leftarrow \mathcal{T}^* V_t \text{ until } \|V^* - V_t\|_\infty \leq \varepsilon$$

- Policy Iteration Modified, where we iterate

$$\begin{cases} V_{t+1} \leftarrow (\mathcal{T}^\pi)^m V_t \\ \pi_{t+1} \leftarrow \arg\max_{a \in \mathcal{A}} (R_a + \gamma \cdot T_a \cdot V^{\pi_t}) \end{cases}$$

[8] [Sutton and Barto, 2018]

MDP solving

In general,

$$\begin{aligned} \text{Solve an MDP} \iff & \text{ Solve } \max_{\pi} V^{\pi} \\ & \text{(Actor-Critic, Deep RL...)} \\ \iff & \text{ Solve } \min_{V \in \mathbb{R}^{\mathcal{S}}} \|V - \mathcal{T}^* V\|_{\infty} \\ & \text{(Dynamic Programming, TD-Learning...)} \end{aligned}$$

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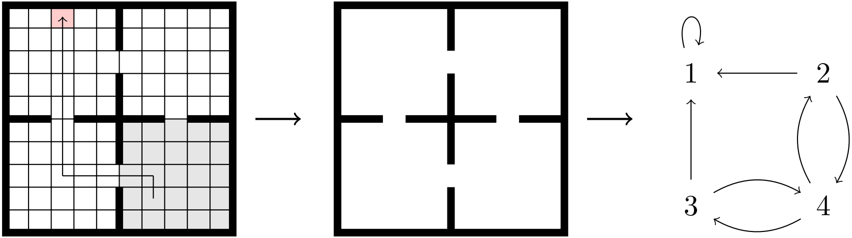
Hierarchical RL — State Abstraction

MDP approximation from state space partition:



Less states, same action space:

$$\mathcal{S} = \bigsqcup_k S_k \to \mathcal{K} = \{s_1, \ldots, s_k\}$$

Averaged transition and reward:

$$T \to \omega \cdot T \cdot \phi$$

$$R \to \omega \cdot R$$

State Abstraction Definition

Definition (Abstract MDP [Li et al., 2006])

Given $\mathcal{M} = (\mathcal{S}, \mathcal{A}, T, R)$ s.t. $\mathcal{S} = \bigsqcup_k S_k$, the abstract MDP

$$(\mathcal{K}, \mathcal{A}, \tilde{T}, \tilde{R})$$

is defined using

- Averaged transition $\tilde{T} = \omega \cdot T \cdot \phi$
- Averaged reward $\tilde{R} = \omega \cdot R$
- Abstract state space $\mathcal{K} = \{s_k\}$

where

- $\omega \in [0, 1]^{K \times \mathcal{S}}$ weights with sum 1 on each region
- $\phi = (\mathbb{1}_{s \in S_k})_{s,k}$

Approximate Bellman Operator

An $L2$ approximation of the optimal Bellman operator

$$\mathcal{T}^* = V \to \max_{a \in \mathcal{A}} \left(R_a + \gamma \cdot T_a \cdot V \right)$$

is its averaged version[9]

$$\Pi \mathcal{T}^*(V) = \phi \cdot \omega \cdot \mathcal{T}^*(V) = \phi \cdot \arg\min_{V_A \in \mathbb{R}^K} \|\phi \cdot V_A - \mathcal{T}^* V\|_2$$

which is less complex to compute!

Moreover, $\tilde{V}^*$ solution of $V = \Pi \mathcal{T}^* V$ checks

$$\|\tilde{V}^* - V^*\|_\infty \leq \max_{1 \leq k \leq K} \frac{\max_{S_k} V - \min_{S_k} V}{1 - \gamma} .$$

[9] [Tsitsiklis and Van Roy, 1996]

Notes on State Abstraction

We note that:

- Grouping similar states limits approximation made[10]:

$$\forall s, s' \in S_k,\ |Q^*(s,a) - Q^*(s',a)| \leq \varepsilon \implies \|V^* - V^{\tilde{\pi}}\|_\infty \leq K \cdot \varepsilon$$

- Efficient partition criterion often depends on V^*, Q^*, π^*...

We propose to:

- Estimate the quality of any valuation of state abstraction
- Refine abstraction along VI steps

[10] [Abel et al., 2016]

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Approximate Bellman Operator and Abstraction

Approximate Bellman $\Pi\mathcal{T}_Q^* = \phi \cdot \omega \cdot \mathcal{T}_Q^*$ is the exact Bellman of an abstract MDP:[11]

Lemma (O.F.)

For any state abstraction $(\mathcal{K}, \omega, \phi)$ *and its value function* $Q_A = \omega \cdot \tilde{Q} \in \mathbb{R}^K$,

$$\phi \cdot \mathcal{T}_{Q,A}^* Q_A = \Pi\mathcal{T}_Q^*(\phi \cdot Q_A)$$

[11] [Forghieri et al., 2024]

Approximate Bellman Operator and Abstraction

Proof.

For any $Q_A \in \mathbb{R}^K$,

$$
\begin{aligned}
\phi \cdot \mathcal{T}^*_{Q,A} Q_A &= \phi \cdot \left(\underline{\mathrm{R}} + \gamma . \underline{\mathrm{T}} \cdot \max_{a \in \mathcal{A}} (Q_A) \right) \\
&= \phi \cdot \left(\omega \cdot R + \gamma \cdot \omega \cdot T \cdot \phi \cdot \max_{a \in \mathcal{A}} (Q_A) \right) \\
&= \phi \cdot \omega \cdot \left(R + \gamma \cdot T \cdot \max_{a \in \mathcal{A}} (\phi \cdot Q_A) \right) \\
&= \Pi \cdot \left(R + \gamma \cdot T \cdot \max_{a \in \mathcal{A}} \left(\tilde{Q} \right) \right) \\
&= \Pi \mathcal{T}^*_Q \tilde{Q}
\end{aligned}
$$

$\square$

Quality of a piecewise constant value function

Theorem (Quality of a piecewise constant value function, O.F.)

Given $\mathcal{M}$*, its abstraction* $(\mathcal{K}, \omega, \phi)$*, and the piecewise constant value function* $\tilde{V}$*,*

$$\|\tilde{V} - V^*\|_\infty \leq \frac{1}{1-\gamma}\left(\max_{1 \leq k \leq K} \operatorname{Span}_{S_k} \mathcal{T}^*\tilde{V} + \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty\right)$$

where $\operatorname{Span}_{S_k} V := \max_{s \in S_k} V(s) - \min_{s \in S_k} V(s)$.

$\rightarrow$ Dependence on the $\tilde{V}$ ($\equiv$ value function on the abstract MDP) and on the aggregation structure !

$\rightarrow$ True for $\mathcal{T}^*_Q$, $\mathcal{T}^\pi$

Quality of a piecewise constant value function

$$\|\tilde{V} - V^*\|_\infty \leq \frac{1}{1-\gamma}\left(\max_{1\leq k\leq K} \operatorname{Span}_{S_k} \mathcal{T}^*\tilde{V} + \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty\right)$$

Two terms:

- $\max_{1\leq k\leq K} \operatorname{Span}_{S_k} \mathcal{T}^*\tilde{V}$: do we lose information aggregating ?
- $\|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty$: is $\tilde{V}$ close to optimal value of abstract MDP ?

Proof of the theorem

Proof.

$$
\begin{aligned}
(1-\gamma)\|V^* - \tilde{V}\|_\infty &\leq \|\tilde{V} - \mathcal{T}^*\tilde{V}\|_\infty \\
&\leq \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty + \|\Pi\mathcal{T}^*\tilde{V} - \mathcal{T}^*\tilde{V}\|_\infty \\
&\leq \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty + \max_k \mathrm{Span}_{S_k} \mathcal{T}^*\tilde{V}
\end{aligned}
$$

□

Quality of a piecewise constant value function

$$\|\tilde{V} - V^*\|_\infty \leq \frac{1}{1-\gamma}\left(\max_{1\leq k\leq K} \operatorname{Span}_{S_k} \mathcal{T}^*\tilde{V} + \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty\right)$$

Fortunately,

- $\max_{1\leq k\leq K} \operatorname{Span}_{S_k} \mathcal{T}^*\tilde{V}$ can decrease refining aggregation $(S_k)_k$
- $\|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty$ can decrease iterating $\Pi\mathcal{T}^*$ over $\tilde{V}$
- $\Pi\mathcal{T}^*$ is simple to compute

Approximate VI is cheaper to compute

Operator	Complexity	Approximation	Complexity
$\mathcal{T}^*$	$\mathcal{S}^2\mathcal{A}$	$\Pi\mathcal{T}^*$	$\mathcal{S}K\mathcal{A}$
$\mathcal{T}^\pi$	$\mathcal{S}^2$	$\Pi\mathcal{T}^\pi$	K^2
$\mathcal{T}_Q^*$	$\mathcal{S}^2\mathcal{A}$	$\Pi\mathcal{T}_Q^*$	$K^2\mathcal{A}$

Table 1: Number of operations necessary to update a value function. $K \ll S$ generally.

$\rightarrow$ Simpler to compute, contract space with factor γ, but converge to $\tilde{V} \neq V^*$... Need to refine aggregation lowering the span !

Progressive State Space Disaggregation Process

We propose starting with a trivial abstraction:

$$K = 1,\ S_1 = \mathcal{S},\ \tilde{V}_0 = (0)_{s\in\mathcal{S}}$$

Then iterate as follows:

1. Apply $\Pi\mathcal{T}^*$ until $\|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty \leq \epsilon$
2. Refine each region by splitting until

$$\max_{S_k} \mathcal{T}^* V_t - \min_{S_k} \mathcal{T}^* V_t \leq \varepsilon$$

   holds for each region k.

$\rightarrow$ This process converges to V^* with arbitrary accuracy.

Disaggregation process

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First disaggregation step

slides/2024_04_24_semdoc/images/rooms_avi_1.png

Second disaggregation step

slides/2024_04_24_semdoc/images/rooms_avi_2.png

Progressive Disaggregation Convergence

Theorem (Final Abstraction Quality, O.F.)

Considering the final value and abstraction $(V_A, (S_k)_k)$*,*

1. *the process finishes in a finite number of steps.*
2. *the distance to optimal value function checks:*

$$\|\phi \cdot V_A - V^*\|_\infty \leq \frac{2\epsilon}{1-\gamma}$$

Moreover, for any region k*,*

$$\forall s, s' \in S_k,\ |V^*(s) - V^*(s')| \leq \frac{4\epsilon}{1-\gamma}\,.$$

→ Abstraction quality is ensured!

Progressive Disaggregation Convergence

Proof.

Two main arguments :

1. The number of partition strictly increases at each step
2. The bound

$$\|\tilde{V} - V^*\|_\infty \leq \frac{1}{1-\gamma}\left(\max_{1\leq k\leq K} \mathrm{Span}_{S_k} \mathcal{T}^*\tilde{V} + \|\tilde{V} - \Pi\mathcal{T}^*\tilde{V}\|_\infty\right)$$

ensures the claimed final precision.

□

Remarks

Advantages :

- Saving time on $\Pi\mathcal{T}^*$ iterations
- Final Abstraction sometimes smaller than original mdp : $K \ll |\mathcal{S}|$
- Convergence guarantee !

Risks :

- Too many disaggregation steps (maximum $|\mathcal{S}|$)
- The final abstraction could be the original MDP itself !

Performance Evaluation

Solving an MDP depends on

- Its structure ($|\mathcal{S}|$, $|\mathcal{A}|$, density of the transition matrix...)
- Wanted final precision to approximate V^* ($\varepsilon = 10^{-3} \nRightarrow \pi = \pi^*$...)
- Chosen discount γ and expected length of the trajectory ($\gamma \ll 1 \iff$ Value Iteration $\gg$ Policy Iteration)

$\rightarrow$ We compare algorithm on the runtime ensuring the same final precision

Toys models

- Four Rooms [Hengst, 2012]
- Racetrack [Sutton and Barto, 2018]

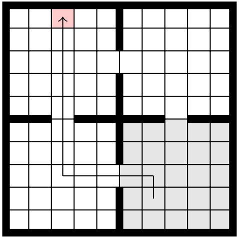

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Figure 5: Four Rooms and Sutton's racectrack models

Real-world models

- Servers in tandem and stochastic arrivals [Tournaire et al., 2022]
- Hydro-valley electricity production management [Carpentier et al., 2018]
- Inventory Control [Winston, 2004]

slides/2024_09_16_argo/images/damm.png

Solving methods

Traditional Dynamic Programming

- Value Iteration
- Modified Policy Iteration

Our (Progressive Disaggregation) adapted to

- Value Iteration, Q-Value Iteration
- Modified Policy Iteration

Alternative Aggregation approach

- Modified Policy Iteration with adaptive aggregation boosting [Bertsekas et al., 1988]
- Aggregation-disaggregation for Temporal-Difference learning [Chen et al., 2022]
- Work in Progress: Abstraction building and solving [Ciosek and Silver, 2015], no precision guarantee

Runtime comparison

$\lvert\mathcal{S}\rvert$	Rooms 193.6k	Barto 1M	Tandem 19.6k	Hydro-valley 118k	Inventory 250k
PDVI	11712.5	73501.4	979.4	>24h	2877.4
PDQVI	**160.3**	**46.8**	657.4	>24h	**1458.5**
VI	2520.2	46524.7	553.2	>24h	4032.7
Chen	12844.1	33238.3	>24h	>24h	>24h
PDPIM	>24h	18044.8	**254.2**	**2051.0**	>24h
PIM	>24h	20164.2	348.8	2273.9	>24h
Bertsekas	>24h	33238.3	>24h	>24h	>24h

Table 2: Runtimes for solving different models. Discount $\gamma = 0.9999$, final precision $\varepsilon = 10^{-3}$.

→ Reasonable gain of time for real-world models.
→ Necessity of a very high discount!

Remarks

Experimental Results:

- For toy models, a suitable decomposition generally exists $\rightarrow$ runtime reduced by a factor of 15
- For real-world models: runtime reduced by a factor of 1.3

Limitations:

- Modest improvements for real-world models with non-smooth optimal value functions: leads to trivial abstraction almost immediately
- For maze-like models, state space exploration is slow, with no improvement in Value Iteration or Policy Iteration Methods

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Conclusion

In those slides:

- Hierarchical RL context
- Link between **Approximate DP and State Abstraction**
- Practical **State Abstraction discovery** criterion
- MDP **solving benchmark**

Upcoming work :

- Total reward convergence proof
- Larger benchmark
- How to transpose it to model free ?

Thank you for listening!

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